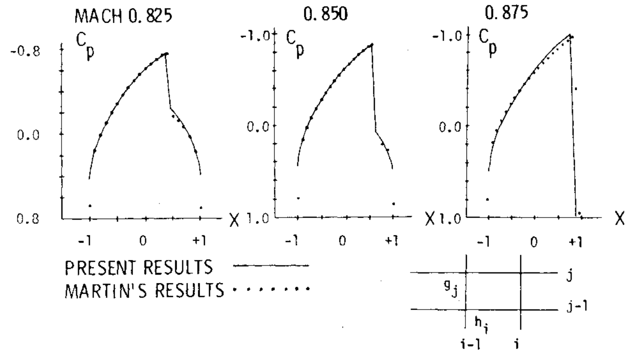

Fig. 1 Surface pressure comparison for 10% thick parabolic arc at sweep 200.

It is useful to examine how the present algorithm fits into the scheme of existing methods. For example, Taylor expansion of the hyperbolic difference form of the usual inviscid equation leads to a differential equation that locally behaves like the VTE, the truncation error term being similar to the viscous diffusion term. (A fully conservative, mixed differencing scheme, including parabolic and shock-point forms of the equation, is given by Murman.[4]) All of these forms can be expanded in Taylor series and can be shown to produce a term analogous to the viscous diffusion term in the truncation error, except that the elliptic and parabolic operators can be regarded as having a zero viscosity coefficient to within second-order accuracy. The key difference between these and the present method is the manner in which diffusion is added to the inviscid equation: a positive ϵ is used everywhere, which formally eliminates the need for mixed differencing. (Thus, one difference formula only is used along a row.) In the numerical work, ϵ is chosen at convenience, for stability reasons. (This does not affect the magnitude of the required jumps, although it does affect the shock thickness obtained.) In this sense, the viscosity is an artificial one, although, as noted, the VTE itself is obtainable through a formal limiting procedure applied to the full viscous equations. The foregoing nonlinear algebraic equations, arranged sequentially with i, are now solved by Newton's iteration with quadratic convergence. The corresponding derivative matrix can be shown to contain one upper codiagonal resting on a lower triangular matrix, with all matrix elements being nonzero. Thus, rapid triangularization is possible, choosing as pivots the successive elements of the upper diagonal, followed by the usual solution by direct elimination. The latest updated values of W are always used in the relaxation procedure, the computations are initialized with $W=0$, and, in the results that follow, we assume that $U_{I,j}=\varphi_{I,j}=W_{i\max,j}=\varphi_{i,j\max}=0$.

Sample Calculations

Calculations carried out for a symmetrical, nonlifting, supercritical 10% thick parabolic arc airfoil are compared with some results of Martin.[3] Twenty uniform grids were assumed over the airfoil and ten for each direction off the airfoil. These off-airfoil horizontal spacings were stretched by a factor of 1.2 each successive grid. Twenty vertical grids were taken. The first two were 2% of chord, and the remainder were stretched by the same 1.2 factor. This resulted in an approximate 4×3 chord computation box. For $\gamma=1.4$ and $\tau=0.05$, three values of M_∞ were considered: 0.825, 0.850, and 0.875, which correspond, respectively, to $K=1.696$, 1.417, and 1.151. The assumed viscosity was 0.0175 throughout, and values of the surface pressure coefficient were obtained by second-order differences.

The calculations were performed on the CDC 7600, and convergence was achieved after 200-250 sweeps of the flowfield. (In some cases, the computations were carried out to 1000 iterations, with only 2-3% fluctuations in surface C_p values.) Figure 1 shows the good qualitative agreement achieved. Numerical experiments showed that increased values of viscosity tended to "smear" the resulting shocks, as expected. The time required per sweep in the relaxation procedure was 0.066 seconds (this is seven times faster than the original algorithm[2]) and is approximately independent of sweep number. The present method also requires only half as many sweeps for convergence, possibly because W is a smoother dependent variable. The shape of the C_p curve and the shock position appeared to be well established by the 150th sweep.

Summary and Conclusions

The scheme presented in this Note introduces artificial viscosity everywhere and formally eliminates the need for mixed differencing. (This idea was first given independently in Ref. 5.) The small disturbance equation used has the form of the VTE, although artifically large values of the viscosity are needed for numerical stability; thus, the computed results, with $\epsilon=O(h)$, are only first-order accurate for the inviscid solution. Also, because of the large resulting matrices, the method is not as efficient as current methods.

Acknowledgments

The author wishes to thank the reviewers and the editor for clarifying some misleading remarks made in an earlier draft of this Note, and his colleagues at Boeing and at Flow Research for many stimulating discussions.

References

[1] Sichel, M., "Theory of Viscous Transonic Flow—A Survey," *AGARD Conference Proceedings, No. 35, Transonic Aerodynamics*, Paper 10, Sept. 1968.

[2] Chin, W. C., "Numerical Solution for Viscous Transonic Flow," *AIAA Journal*, Vol. 15, Sept. 1977, pp. 1360-1362.

[3] Martin, E. D., "A Fast Semi-direct Method for Computing Transonic Aerodynamic Flows," *AIAA 2nd Computational Fluid Dynamics Conference Proceedings*, Hartford, Conn., June 1975, pp. 162-174.

[4] Murman, E. M., "Analysis of Embedded Shock Waves Calculated by Relaxation Methods," *AIAA Journal*, Vol. 12, May 1974, pp. 626-633.

[5] Bauer, F., Garabedian, P., Korn, D., and Jameson, A., "Supercritical Wing Sections II," *Lecture Notes in Economics and Mathematical Systems*, Vol. 108, Springer-Verlag, Berlin, 1975.

Notes on the Flow Near a Wall and Dividing Streamline Intersection

B. S. Dandapat* and A. S. Gupta†
Indian Institute of Technology, Kharagpur, India

THE flow of an incompressible viscous fluid with negligible inertia forces near a corner between two plane boundaries was discussed by Moffatt.[1] It is found that, when either or both of the boundaries is rigid and the corner angle is less than a critical value, the flow consists of a sequence of eddies of decreasing size and rapidly decreasing intensity as the corner is approached. Detailed investigations of two-dimensional corner flow also were made by Lugt and Schwiderski.[2]

Received Dec. 13, 1977; revision received May 5, 1978.

Index category: Viscous Nonboundary-Layer Flows.

*Research Assistant, Mathematics Department.
†Professor, Mathematics Department.

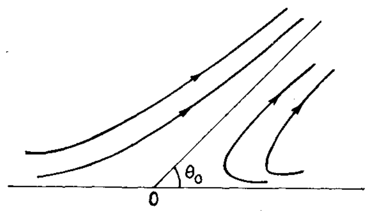

Fig. 1 Flow near a dividing streamline.

Batchelor,[3] on the other hand, gave a solution for the Stokes flow in the neighborhood of a point of zero tangential stress at a plane rigid boundary (Fig. 1). In his solution, the dividing streamline that emanates from 0 (the point of zero friction) makes, with the wall, a given angle θ_0, although Batchelor did not discuss how this angle was to be determined. Schubert[4] attempted to determine this angle by using the conditions of zero velocity normal to the dividing streamline $\theta=\theta_0$, along with the continuity of tangential and normal stress on $\theta=\theta_0$ and no-slip conditions on $\theta=0$ and $\theta=\pi$. However, the normal stress condition used by Schubert is in error, and his entire analysis is therefore of doubtful validity. The aim of this Note is to give a correct solution to this problem and deduce Batchelor's solution as a very special case. The configuration of the present problem arises in several interesting flows. The motion of a viscous fluid over an obstacle lying on a wall can be expected to give rise to a recirculating region bounded by the obstacle, the wall, and a dividing streamline.

We use plane polar coordinates (γ,θ) with origin at 0 (Fig. 1). Introducing the stream function $\psi(\gamma,\theta)$, the velocity components along the radial and tangential directions are

$$v_\gamma=\frac{1}{\gamma}\frac{\partial\psi}{\partial\theta},\quad v_\theta=-\frac{\partial\psi}{\partial\gamma} \tag{1}$$

In the steady flow, the stream function ψ satisfies the Stokes equation $\nabla^4\psi=0$, which admits of separable solution of the form $\psi\sim\gamma^\lambda f_\lambda(\theta)$. As pointed out by Moffatt, sufficiently near the intersection of the wall and the dividing streamline inertial terms in the Navier-Stokes equations can be neglected, and the flow is represented adequately by the Stokes equation, provided that $Re(\lambda)>0$. The solutions of $\nabla^4\psi=0$ in regions 1 and 2 are given by

$$\psi=\gamma^\lambda[A_1\cos\lambda\theta+B_1\sin\lambda\theta+C_1\cos(\lambda-2)\theta+D_1\sin(\lambda-2)\theta],\quad 0\le\theta\le\theta_0 \tag{2}$$

$$\psi=\gamma^\lambda[A_2\cos\lambda\theta+B_2\sin\lambda\theta+C_2\cos(\lambda-2)\theta+D_2\sin(\lambda-2)\theta],\quad \theta_0\le\theta\le\pi \tag{3}$$

respectively, provided that $\lambda\neq 2$. The case $\lambda=2$ will not be relevant to the present problem, as will be shown later. It follows from Eqs. (1) that the velocity components are proportional to $\gamma^{\lambda-1}$, and, since velocity must vanish as $\gamma\to 0$, we must have $Re(\lambda)>1$.

The boundary conditions are

$$v_{\gamma 1}(\theta=0)=v_{\theta 1}(\theta=0)=0 \tag{4}$$

$$v_{\gamma 2}(\theta=\pi)=v_{\theta 2}(\theta=\pi)=0 \tag{5}$$

$$v_{\theta 1}(\theta=\theta_0)=v_{\theta 2}(\theta=\theta_0)=0 \tag{6}$$

$$(\partial v_{\gamma 1}/\partial\theta)_{\theta=\theta_0}=(\partial v_{\gamma 2}/\partial\theta)_{\theta=\theta_0} \tag{7}$$

$$\left(-p_1+2\mu\left[\frac{1}{\gamma}\frac{\partial v_{\theta 1}}{\partial\theta}+\frac{v_{\gamma 1}}{\gamma}\right]\right)_{\theta=\theta_0}=\left(-p_2+2\mu\left[\frac{1}{\gamma}\frac{\partial v_{\theta 2}}{\partial\theta}+\frac{v_{\gamma 2}}{\gamma}\right]\right)_{\theta=\theta_0} \tag{8}$$

where the subscripts 1 and 2 refer to regions 1 and 2, respectively. The conditions (4-6) are self-explanatory, whereas conditions (7) and (8) express the continuity of the tangential and the normal stress on the dividing streamline $\theta=\theta_0$. It may be noted that the normal stress condition on $\theta=\theta_0$ used by Schubert is erroneous [cf. Eq. (11) of his paper].

Boundary conditions (4-8), when applied to Eqs. (2) and (3), yield the following set of homogeneous equations for the arbitrary constants A_i, B_i, C_i, and D_i $(i=1,2)$:

$$B_1\lambda+D_1(\lambda-2)=0 \tag{9}$$

$$A_1+C_1=0 \tag{10}$$

$$-A_2\lambda\sin\lambda\pi+B_2\lambda\cos\lambda\pi-(\lambda-2)C_2\sin(\lambda-2)\pi+D_2(\lambda-2)\cos(\lambda-2)\pi=0 \tag{11}$$

$$A_2\cos\lambda\pi+B_2\sin\lambda\pi+C_2\cos(\lambda-2)\pi+D_2\sin(\lambda-2)\pi=0 \tag{12}$$

$$A_1\cos\lambda\theta_0+B_1\sin\lambda\theta_0+C_1\cos(\lambda-2)\theta_0+D_1\sin(\lambda-2)\theta_0=0 \tag{13}$$

$$A_2\cos\lambda\theta_0+B_2\sin\lambda\theta_0+C_2\cos(\lambda-2)\theta_0+D_2\sin(\lambda-2)\theta_0=0 \tag{14}$$

$$\lambda^2A_1\cos\lambda\theta_0+\lambda^2B_1\sin\lambda\theta_0+(\lambda-2)^2C_1\cos(\lambda-2)\theta_0+D_1(\lambda-2)^2\sin(\lambda-2)\theta_0-\lambda^2A_2\cos\lambda\theta_0-\lambda^2B_2\sin\lambda\theta_0-(\lambda-2)^2C_2\cos(\lambda-2)\theta_0-(\lambda-2)^2D_2\sin(\lambda-2)\theta_0=0 \tag{15}$$

$$A_1\lambda\sin\lambda\theta_0-B_1\lambda\cos\lambda\theta_0+(\lambda+2)C_1\sin(\lambda-2)\theta_0-(\lambda+2)D_1\cos(\lambda-2)\theta_0-A_2\lambda\sin\lambda\theta_0+B_2\lambda\cos\lambda\theta_0-(\lambda+2)C_2\sin(\lambda-2)\theta_0+(\lambda+2)D_2\cos(\lambda-2)\theta_0=0 \tag{16}$$

While deriving Eq. (16) from Eq. (8), the momentum equations (with inertial terms neglected) have been used to eliminate the pressure term. Since eddying motions are not considered, λ is assumed to be real in the present analysis. The eigenvalue relation connecting λ and θ_0 is obtained by setting the 8×8 coefficient determinant of Eqs. (9-16) equal to zero. After a lengthy calculation, it is found that

$$\{-\lambda^2+\cos[2(\lambda-1)\theta_0]+(\lambda^2-1)\cos 2\theta_0\}\times\{(\lambda-1)^2\sin 2\theta_0+(1-\lambda)\sin[2(\lambda-1)(\theta_0-\pi)]\}+\{(1-\lambda)\sin[2(\lambda-1)\theta_0]+(\lambda-1)^2\sin 2\theta_0\}\times\{\lambda^2-(\lambda^2-1)\cos 2\theta_0-\cos[2(\lambda-1)(\theta_0-\pi)]\}=0 \tag{17}$$

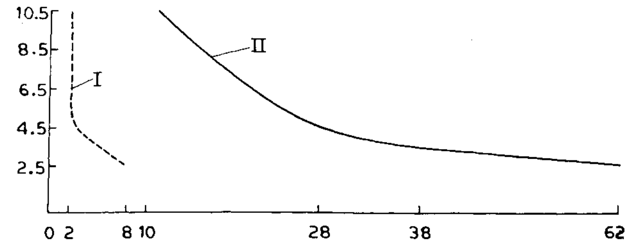

Fig. 2 The eigenvalues λ vs θ_0. The scale for the curve I is one unit = 0.1 deg and for the curve II is one unit = 1 deg.

Since $Re(\lambda) > 1$ and $\lambda \neq 2$ and the shear stress vanishes at 0, in what follows we confine ourselves to all real values of $\lambda > 2$. It can be seen easily that, for any integer value of $\lambda \geq 3$, Eq. (17) is satisfied identically by any value of θ_0. In fact, Batchelor's solution corresponds to $\lambda = 3$. Our analysis, however, reveals that it also is possible to have a dividing streamline with arbitrary θ_0 for any integer value of $\lambda > 3$. Furthermore, when $\lambda > 2$ and is not an integer, θ_0 is determined by solving Eq. (17) numerically. For a given value of λ, Eq. (17) is a transcendental equation in θ_0, which admits of infinite number of roots. The two curves in Fig. 2 correspond to values of λ for the first two positive roots. It can be seen that, as λ increases, θ_0 decreases. We thus find that an infinite number of solutions is possible for flow near a dividing streamline. The uniqueness of the solution is perhaps achieved by considering the circumstances outside the region in which each of the preceding solutions is valid.

We have seen from Eq. (17) that any integer value of $\lambda \geq 3$ is an eigenvalue regardless of the value of θ_0. It is of interest to see what happens when $\lambda = 2$. Schubert investigated this case also and arrived at the erroneous conclusion that $\lambda = 2$ is an eigenvalue when $\theta_0 = \pi/4$. We now show that $\lambda = 2$ does not correspond to a solution with a dividing streamline. In this case, the solutions of $\nabla^4 \psi = 0$ in the two regions are

$$\psi_i = \gamma^2 (A_i \cos 2\theta + B_i \sin 2\theta + C_i \theta + D_i) \quad (i = 1, 2) \qquad (18)$$

When Eq. (18) is used in the boundary conditions (4-8) and the constants A_i, B_i, C_i, and D_i ($i = 1,2$) are eliminated, the following equation is obtained:

$$(1 - \cos 2\theta_0)(2 \log\gamma \cdot \cos 2\theta_0 - 1 + \cos 2\theta_0) = 0 \qquad (19)$$

which can hold only when $\cos 2\theta_0 = 1$, leading to $\theta_0 = n\pi$, $n = 0, 1, 2, \ldots$. Thus when $\lambda = 2$ no solution with a dividing streamline is possible, contrary to the conclusion of Schubert. This result also could have been expected from physical considerations, since, when $\psi \sim \gamma^2 f_2(\theta)$, the expression for the shear stress becomes independent of γ and hence cannot vanish at the point 0 (Fig. 1), which contradicts the fact that 0 is a point of zero skin friction.

Acknowledgment

The first author acknowledges the financial assistance provided by the Council of Scientific and Industrial Research, Delhi, for carrying out this research.

References

[1] Moffatt, H. K., "Viscous and Resistive Eddies near a Sharp Corner," *Journal of Fluid Mechanics*, Vol. 18, 1964, pp. 1-18.

[2] Lugt, H. J. and Schwiderski, E. W., "Flows around Dihedral Angles," *Proceedings of the Royal Society, (London)*, Vol. A285, 1965, pp. 382-399.

[3] Batchelor, G. K., *An Introduction to Fluid Dynamics*, Cambridge University Press, London, 1967, pp. 226-227.

[4] Schubert, G., "Flow near the Intersection of a Wall and a Dividing Streamline," *AIAA Journal*, Vol. 6, March 1968, pp. 549-550.

Simplified Formulas for Lift and Moment in Unsteady Thin Airfoil Theory

Nelson H. Kemp*
Physical Sciences Inc., Woburn, Mass.

RECENTLY, Williams[1] presented results for the pressure distribution, lift, and moment of an arbitrary oscillatory motion in subsonic thin airfoil theory. He showed that they could all be obtained from a knowledge of the loading distributions for plunging (heaving) motion and pitching about the leading edge. He expressed his results in a coordinate system originating at the leading edge, and found the lift to depend only on the plunging loading, while the moment depends on both the plunging and pitching loading.

The purpose of this Note is to point out that the formula for the moment takes a remarkably simple form when reformulated in a coordinate system originating at the airfoil center, with pitching taken about that center. The moment then depends only on the loading distribution for pitching, just as the lift depends only on the loading distribution for plunging.

Williams uses a set of dimensionless variables in which length is measured in chord lengths c, velocities in stream speed U, time in units of c/U, loading P in units of ρU^2, lift in units of $\rho U^2 c$, and moment in units of $\rho U^2 c^2$. The dimensionless (reduced) frequency k is related to the frequency ν by $k = \nu c/U$. He denotes by P_h the loading induced by the plunging downwash ik, and by P_α the loading induced by the pitching downwash (about the leading edge) $1 + ikx$. In those terms, he finds the lift for an arbitrary downwash $w(x)$ to be his Eq. (18):

$$L = \int_0^1 w(1-x) P_h(x) \frac{\mathrm{d}x}{ik} \qquad (1)$$

The moment about the leading edge (nose down) is given by his Eq. (19):

$$M = -\int_0^1 w(1-x) \left[(1+ik) P_h(x) - ik P_\alpha(x) \right] \frac{\mathrm{d}x}{k^2} \qquad (2)$$

In computing the aerodynamic loads on oscillating airfoils, it is often convenient to use a coordinate system originating at the airfoil center, normalized by the half-chord. If this variable is called $\bar{x}$, it is related to x by

$$\bar{x} = 2x - 1 \qquad -1 < \bar{x} < 1 \qquad (3)$$

Any function of x can be expressed as a function of $\bar{x}$ by

$$f(x) = f[(\bar{x}+1)/2] = \bar{f}(\bar{x}) = \bar{f}(2x-1) \qquad (4a)$$

$$f(1-x) = \bar{f}(2-2x-1) = \bar{f}(1-2x) = \bar{f}(-\bar{x}) \qquad (4b)$$

We now define $P^{(0)}$ as the loading induced by unit plunging downwash, so that

$$P^{(0)}(x) = \bar{P}^{(0)}(\bar{x}) = P_h(x)/ik \qquad (5)$$

Received March 20, 1978.
Index categories: Nonsteady Aerodynamics.

*Principal Scientist. Associate Fellow AIAA.